Let $U \sim \chi^2_m$, $V \sim \chi^2_n$, and U and V are independent.
Then the distribution of $\dfrac{U/m}{V/n}$ ~~has~~ is defined as $F_{m,n}$ — the F distribution with $\boxed{m}$ numerator degrees of freedom and $\boxed{n}$ denominator degrees of freedom.

Theorem 4 Let $X_1, \cdots, X_n \overset{iid}{\sim} N(\mu_X, \sigma_X^2)$, $Y_1, \cdots, Y_m \overset{iid}{\sim} N(\mu_Y, \sigma_Y^2)$
$X_1, \cdots, X_n, Y_1, \cdots, Y_m$ are independent
~~Then~~ Let $S_X^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2$, $S_Y^2 = \frac{1}{m-1}\sum_{i=1}^{m}(Y_i - \bar{Y}_m)^2$

Then $\dfrac{S_X^2/\sigma_X^2}{S_Y^2/\sigma_Y^2} \sim F_{n-1,m-1}$.

Pf: Recall that (Theorem 2): $\dfrac{(n-1)S_X^2}{\sigma_X^2} \sim \chi^2_{n-1}$, $\dfrac{(m-1)S_Y^2}{\sigma_Y^2} \sim \chi^2_{m-1}$

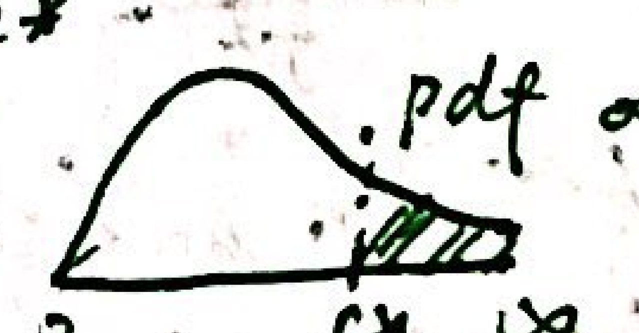
Since $X_1, \cdots, X_n, Y_1, \cdots, Y_m$ are independent, S_X^2 and S_Y^2 are independent
So by the definition of F distribution,

$$\frac{\dfrac{(n-1)S_X^2}{\sigma_X^2}\Big/(n-1)}{\dfrac{(m-1)S_Y^2}{\sigma_Y^2}\Big/(m-1)} = \frac{S_X^2/\sigma_X^2}{S_Y^2/\sigma_Y^2} \sim F_{n-1,m-1}.$$

Application: two-sample F test for equal variance
$H_0: \sigma_X^2 = \sigma_Y^2$ $\quad H_1: \sigma_X^2 > \sigma_Y^2$

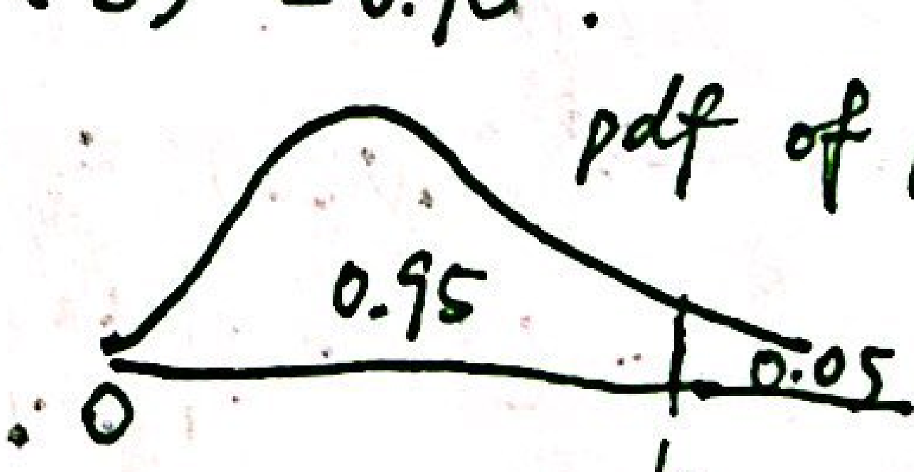
Test statistic: $F = \dfrac{S_X^2}{S_Y^2} \overset{\text{under } H_0}{=\!=} \dfrac{S_X^2/\sigma_X^2}{S_Y^2/\sigma_Y^2} \sim F_{n-1,m-1}$

Say F has observed value f^*
p-value $= \mathbb{P}_{H_0}(F \geq f^*) =$



Example: Two independent samples from two normal distributions w/ the same variance. The sample sizes are $n_1 = 6$, $n_2 = 10$.
Denote their sample variances by S_1^2 and S_2^2.
Find constant b such that $\mathbb{P}\left(\frac{S_1^2}{S_2^2} < b\right) = 0.95$.

Ans: $\dfrac{S_1^2}{S_2^2} \underset{\sigma_1^2 = \sigma_2^2}{=} \dfrac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2} \sim F_{n_1-1, n_2-1} = F_{5,9}$



So $b = 3.48$ so that $P\left(\frac{S_1^2}{S_2^2} < b\right) = 0.95$

1. If $X \sim t_n$, then $X^2 \sim F_{1,n}$ (analogous to $Z \sim N(0,1) \Rightarrow Z^2 \sim \chi^2_1$)

Pf: Given the definition of t_n, we can let $Z \sim N(0,1)$ and $U \sim \chi^2_n$ be independent, then $\frac{Z}{\sqrt{U/n}} \sim t_n$

Let $X = \frac{Z}{\sqrt{U/n}}$, then $X \sim t_n$

$$X^2 = \frac{Z^2}{U/n} = \frac{Z^2/1}{U/n} \sim F_{1,n} \text{ because } Z^2 \sim \chi^2_1 \quad \square$$

which follows the definition of $F_{1,n}$

2. ~~Let~~ If $X \sim F_{m,n}$, then $\frac{1}{X} \sim F_{n,m}$

Clarification: about CLT: $X_1, \cdots, X_n \overset{iid}{\sim} \mu, \sigma^2 < \infty$

when n is large: $\sum_{i=1}^{n} X_i \overset{approx}{\sim} N(n\mu, n\sigma^2)$

$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i \overset{approx}{\sim} N(\mu, \frac{\sigma^2}{n})$ $\Rightarrow \frac{\sum X_i - n\mu}{\sqrt{n}\sigma} = \frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \overset{approx}{\sim} N(0,1)$

Practices: $\underbrace{Z_1, \cdots, Z_{16} \overset{iid}{\sim} N(0,1), \ X_1, \cdots, X_{64}}_{\text{independent}} \overset{iid}{\sim} N(\mu, 1)$

a) Find $P(Z_1 > 2) \approx 0.02275$

b) Find $P(\sum_{i=1}^{16} Z_i > 2)$

Ans: $\sum_{i=1}^{16} Z_i \sim N(0, 16) \Rightarrow \frac{\sum Z_i}{4} \sim N(0,1)$

$P(\sum_{i=1}^{16} Z_i > 2) = 0.3085$

or $P\left(\frac{\sum_{i=1}^{16} Z_i}{4} > \frac{2}{4}\right) = 0.3085$

c) Find $P(\sum_{i=1}^{16} Z_i^2 > 6.91)$

Ans: $\sum_{i=1}^{16} Z_i^2 \sim \chi^2_{16}$

$P(\sum_{i=1}^{16} Z_i^2 > 6.91) = 0.975$

d) Find c s.t. $P(S^2 > c) = 0.05$ where $S^2 = \frac{1}{16-1}\sum_{i=1}^{16}(Z_i - \bar{Z}_{16})^2$

Ans: $\frac{(16-1) \cdot S^2}{1} = 15S^2 \sim \chi^2_{15}$

$P(S^2 > c) = P(15S^2 > 15c) = 0.05 \Rightarrow 15c = 24.99579 \Rightarrow c = 1.666$

e) Find the distribution of $Y = \sum_{i=1} Z_i + \sum_{i=1} (X_i - \mu)^2$

Ans: $Z_1, \cdots, Z_{16}, X_1 - \mu, \cdots, X_{64} - \mu \overset{iid}{\sim} N(0,1)$

so $Y \sim \chi^2_{80}$

f) Find $E[Y] = 80$

g) Find $var[Y] = 2 \times 80 = 160$

h) Find $P(Y > 105)$ and approx it by CLT.

Ans: exact: $Y \sim \chi^2_{80} \Rightarrow P(Y > 105) = 0.0319$

approx by CLT: $Y \overset{approx}{\sim} N(80, 160)$

$P(Y > \cancel{100}\ 105) = 0.024$

or $P\left(\frac{Y-80}{\sqrt{160}} > \frac{\cancel{100}\ 105 - 80}{\sqrt{160}} = \frac{\cancel{20}\ 25}{4\sqrt{10}} = \cancel{\ \ }\right)$

i) Find c such that

$$\frac{c \sum_{i=1}^{16} Z_i^2}{\sum_{i=1}^{64} (X_i - \mu)^2} \sim F_{16, \cancel{80}\, 64}$$

Ans: by the definition of $F_{16,80}$, $\sum_{i=1}^{16} Z_i^2 \sim \chi^2_{16}$, $\cancel{Y \sim \chi^2_{80}}$

$\sum_{i=1}^{64} (X_i - \mu)^2 \sim \chi^2_{64}$, and $\sum Z_i^2$ and $\sum (X_i - \mu)^2$ are independent

So $$\frac{\sum_{i=1}^{16} Z_i^2 / 16}{\sum_{i=1}^{64} (X_i - \mu)^2 / 64} \sim F_{16, 64}$$

so $c = 4$.

j) Let $Q \sim \chi^2_{60}$ and independent of $Z_1 \sim N(0,1)$. Find c s.t

$P\left(\frac{Z_1}{\sqrt{Q}} < c\right) = 0.95$

Ans. by the definition of t dist:

$$\frac{Z_1}{\sqrt{Q/60}} \sim t_{60}$$

$P\left(\frac{Z_1}{\sqrt{Q}} < c\right) = P\left(\frac{Z_1}{\sqrt{Q/60}} < \sqrt{60} \cdot c\right) = 0.95$

$\Rightarrow \sqrt{60} \cdot c = 1.67 \Rightarrow c = 0.2157$