Special case ③ of Gamma(α, β): $\alpha = \frac{n}{2}$, $\beta = 2$ — χ_n^2 dist

Recall: the MGF of Gamma(α,β) is $M(t) = (1-\beta t)^{-\alpha}$.

so the MGF of χ_n^2 is $M(t) = (1-2t)^{-\frac{n}{2}}$

-- χ_1^2 is $M(t) = (1-2t)^{-1/2}$

Theorem 2: Let $Z_1, \cdots, Z_n \overset{iid}{\sim} N(0,1)$, then $\sum_{i=1}^{n} Z_i^2 \sim \chi_n^2$.

Pf: Based on Theorem 1, $Z_i^2 \sim \chi_1^2$, so the MGF of Z_i^2 is

$$M_{Z_i^2}(t) = (1-2t)^{-\frac{1}{2}}$$

Because $Z_1^2, \cdots, Z_n^2$ are independent

$$M_{\sum_{i=1}^n Z_i^2}(t) = M_{Z_1^2}(t) \cdots M_{Z_n^2}(t) = \prod_{i=1}^{n} M_{Z_i^2}(t)$$

$$= \left((1-2t)^{-\frac{1}{2}}\right)^n = (1-2t)^{-\frac{n}{2}} = \text{MGF of } \chi_n^2$$

so $\sum_{i=1}^{n} Z_i^2 \sim \chi_n^2 = \text{Gamma}(\frac{n}{2}, 2)$

Corollary: Let $X_1, \cdots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$. then $\sum_{i=1}^{n} \left(\frac{X_i - \mu}{\sigma}\right)^2 \sim \chi_n^2$.

Pf: Since $X_i \sim N(\mu, \sigma^2)$, so $\frac{X_i - \mu}{\sigma} \sim N(0,1)$. Then apply Thm 2.

Theorem 3: Let $X \sim \chi_m^2$, $Y \sim \chi_n^2$, and X & Y are independent. then

$$X + Y \sim \chi_{m+n}^2$$

Pf: The MGF of X is $M_X(t) = (1-2t)^{-\frac{m}{2}}$

$\cdots\cdots$ Y is $M_Y(t) = (1-2t)^{-\frac{n}{2}}$

Because X & Y are independent, the MGF of $X+Y$ is

$$M_{X+Y}(t) = M_X(t) \cdot M_Y(t) = (1-2t)^{-\frac{m}{2}} \cdot (1-2t)^{-\frac{n}{2}} = (1-2t)^{-\frac{m+n}{2}}$$

$$= \text{MGF of } \chi_{m+n}^2.$$

So $X + Y \sim \chi_{m+n}^2$.

Recall: $X \sim N(\mu_1, \sigma_1^2)$, $Y \sim N(\mu_2, \sigma_2^2)$, and X, Y are independent.

then $X + Y \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$

$X - Y \sim N(\mu_1 - \mu_2, \sigma_1^2 + \sigma_2^2)$

$aX + bY \sim N(a\mu_1 + b\mu_2, a^2\sigma_1^2 + b^2\sigma_2^2)$

prove them by MGF

Important properties of normal r.v.s.

Theorem 1: Let $X_1, \cdots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$. Define $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$

Show that $\bar{X}_n$ and $X_i - \bar{X}_n$ are independent.

Pf: To show that the joint MGF of $(\bar{X}_n, X_i - \bar{X}_n)$ can be factorized into the MGF of $\bar{X}_n$ and the MGF of $X_i - \bar{X}_n$.

Recall that $M_{(X,Y)}(s,t) = M_X(s) \cdot M_Y(t) \Leftrightarrow X \,\&\, Y$ are indep.

Based on the definition of joint MGF,

$$M_{(\bar{X}_n, X_i - \bar{X}_n)}(s,t) = E\left[e^{s\bar{X}_n + t(X_i - \bar{X}_n)}\right]$$

Note that $s\bar{X}_n + t(X_i - \bar{X}_n) = (s-t)\bar{X}_n + tX_i = (s-t)\frac{1}{n}\sum_{i=1}^{n} X_i + tX_i$

$$= \left(\frac{s-t}{n} + t\right) X_i + \frac{s-t}{n} \cdot \sum_{j \neq i} X_j$$

So
$$M_{(\bar{X}_n, X_i - \bar{X}_n)}(s,t) = E\left[e^{\left(\frac{s-t}{n} + t\right) X_i + \frac{s-t}{n}\sum_{j\neq i} X_j}\right]$$

$$\overset{\cancel{X_1, \cdots, X_n \text{ iid}}}{=} E\left[e^{\left(\frac{s-t}{n}+t\right)X_i} \cdot \prod_{j\neq i} e^{\frac{s-t}{n} X_j}\right]$$

$$\overset{X_1, \cdots, X_n \text{ iid}}{=} E\left[e^{\left(\frac{s-t}{n}+t\right)X_i}\right] \cdot \prod_{j \neq i} E\left[e^{\frac{s-t}{n}X_j}\right]$$

$$= M_{X_i}\left(\frac{s-t}{n} + t\right) \cdot \prod_{j \neq i} M_{X_j}\left(\frac{s-t}{n}\right)$$

Recall that $X_i \sim N(\mu, \sigma^2)$, the MGF is $M_{X_i}(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$

So
$$M_{(\bar{X}_n, X_i - \bar{X}_n)}(s,t) = e^{\mu\left(\frac{s-t}{n}+t\right) + \frac{\sigma^2}{2}\left(\frac{s-t}{n}+t\right)^2} \cdot \underbrace{\prod_{j\neq i} e^{\mu\left(\frac{s-t}{n}\right) + \frac{\sigma^2}{2}\left(\frac{s-t}{n}\right)^2}}_{\text{same term multiplied by } (n-1) \text{ times}}$$

$$= \left(e^{\mu\left(\frac{s-t}{n}\right) + \frac{\sigma^2}{2}\left(\frac{s-t}{n}\right)^2}\right)^{n-1}$$

$$= e^{\mu\left(\frac{s-t}{n}+t\right) + \frac{\sigma^2}{2}\left(\frac{s-t}{n}+t\right)^2 + \mu\left(\frac{s-t}{n}\right)\cdot(n-1) + \frac{\sigma^2}{2}\left(\frac{s-t}{n}\right)^2\cdot(n-1)}$$

Want $M_{(\bar{X}_n, X_i - \bar{X}_n)}(s,t) = M_{\bar{X}_n}(s) \cdot M_{X_i - \bar{X}_n}(t)$. (*)

$$M_{\bar{X}_n}(s) \underset{\bar{X}_n \sim N(\mu, \frac{\sigma^2}{n})}{=} e^{\mu s + \frac{\sigma^2 s^2}{n \cdot 2}}$$

~~$M_{X_i-\bar{X}_n}$~~ Note that X_i and $\bar{X}_n$ are NOT independent

But we can derive the dist of $X_i - \bar{X}_n$:

$$X_i - \bar{X}_n = X_i - \frac{1}{n}\sum_{i=1}^{n} X_i = \left(1-\frac{1}{n}\right)X_i - \frac{1}{n}\sum_{j\neq i} X_j \sim N\left(0, \frac{n-1}{n}\sigma^2\right)$$

$$\begin{cases} E[X_i - \bar{X}_n] = E[X_i] - E[\bar{X}_n] = \mu - \mu = 0 \\ var[X_i - \bar{X}_n] = var\left[\left(1-\frac{1}{n}\right)X_i - \frac{1}{n}\sum_{j\neq i} X_j\right] \overset{ind}{=} \left(1-\frac{1}{n}\right)^2 var[X_i] + \left(\frac{1}{n}\right)^2 \cdot \sum_{j\neq i} var[X_j] \end{cases}$$

$$= \left(1-\frac{1}{n}\right)^2\sigma^2 + \left(\frac{1}{n}\right)^2 (n-1)\sigma^2$$

$$= \left(\frac{(n-1)^2}{n^2} + \frac{n-1}{n^2}\right)\sigma^2$$

$$= \frac{n^2-2n+1+n-1}{n^2}\sigma^2 = \frac{n^2-n}{n^2}\sigma^2$$

$$= \frac{n-1}{n}\sigma^2$$

So $M_{X_i-\bar{X}_n}(t) = e^{\frac{(n-1)\sigma^2 t^2}{n\cdot 2}}$

We are left to show that $(*)$ is equivalent to

$$\mu\left(\frac{s-t}{n}+t\right) + \frac{\sigma^2}{2}\left(\frac{s-t}{n}+t\right)^2 + \mu\left(\frac{s-t}{n}\right)(n-1) + \frac{\sigma^2}{2}\left(\frac{s-t}{n}\right)^2(n-1)$$

$$\overset{?}{=} \mu s + \frac{\sigma^2 s^2}{2n} + \frac{(n-1)\sigma^2 t^2}{2n}$$