Continuity Theorem (conv of MGF $\Rightarrow$ conv of CDF)

Let $X_1, X_2, \cdots$ be a sequence of R.V.s with CDFs $F_1(\cdot), F_2(\cdot), \cdots$ and $M_{X_1}(\cdot), M_{X_2}(\cdot), \cdots$. Suppose that

$$M_{X_n}(t) \to M_X(t) \text{ as } n\to\infty \text{ for all } t\in\mathbb{R}$$

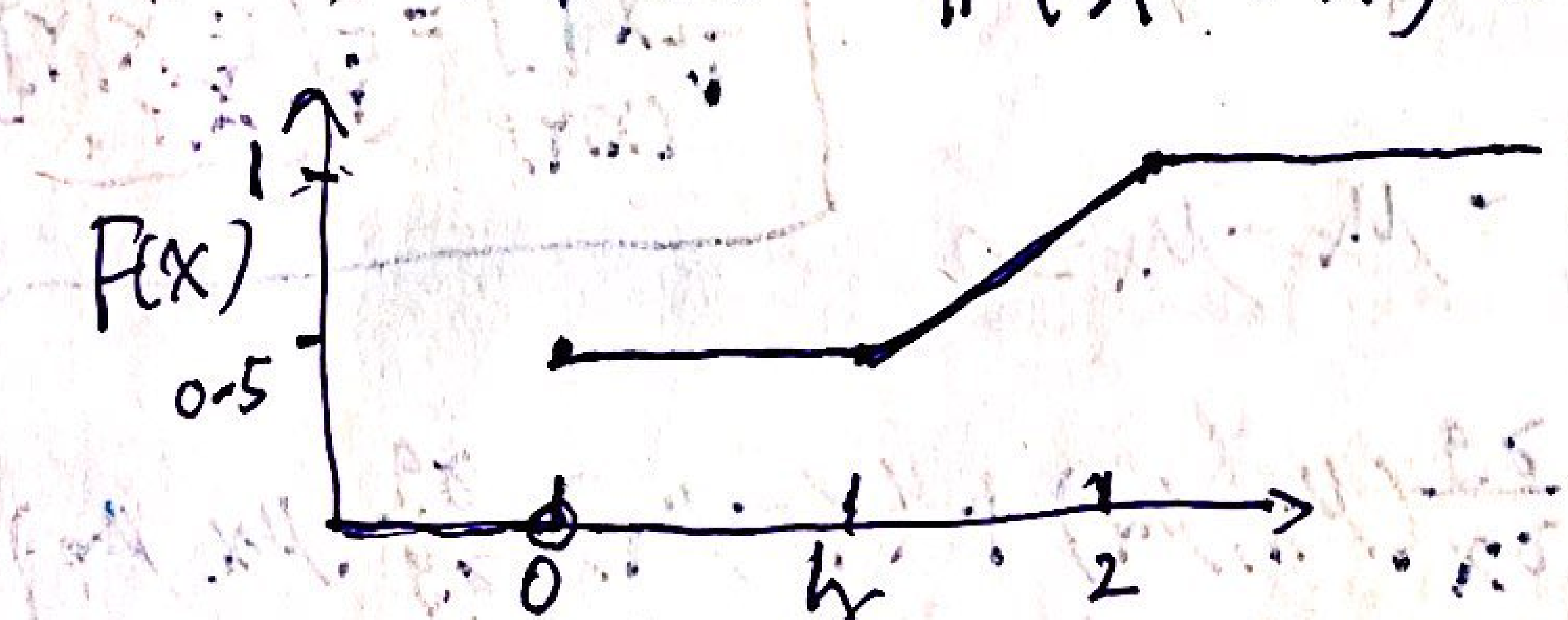
where X is another R.V. with CDF $F(\cdot)$ and MGF $M_X(\cdot)$.

Then $F_n(x) \to F(x)$ as $n\to\infty$ for all x at which F is continuous

---

e.g. $X = 0$ with probability 0.5, & $X \sim \mathrm{Unif}[1,2]$ with probability 0.5

Recall $F(x) = \mathbb{P}(X \le x) = \begin{cases} 0 & \text{if } x<0 \\ 0.5 & x=0 \\ 0.5 & 0<x\le 1 \\ 0.5+0.5(x-1) & 1<x\le 2 \\ 1 & x>2 \end{cases}$



$\mathbb{P}(X\le 0.5) = \mathbb{P}(X=0)$

$\mathbb{P}(X\le x) = \mathbb{P}(X=0) + \mathbb{P}(1<X\le x) = 0.5 + 0.5\cdot(x-1)$

F is cont. everywhere except at 0.

Example: Poisson(λ) converges to a normal dist as $\lambda\to\infty$.

Pf: Let $0<\lambda_1<\lambda_2<\cdots<\lambda_n<\cdots$ be an increasing sequence $\lambda_n\to\infty$ as $n\to\infty$

Let $X_n \sim$ Poisson(λ_n). $E[X_n]=\lambda_n$; $var[X_n]=\lambda_n$

Recall X_n has MGF $M_{X_n}(t) = e^{\lambda_n(e^t-1)}$ $t\in\mathbb{R}$

Standardize X_n as

$$Z_n = \frac{X_n-\lambda_n}{\sqrt{\lambda_n}} = -\sqrt{\lambda_n} + \left(\frac{1}{\sqrt{\lambda_n}}\right)X_n$$

Try to show that Z_n converges to a $N(0,1)$ R.V. as $\lambda_n\to\infty$ ($n\to\infty$)

So $M_{Z_n}(t) = e^{-\sqrt{\lambda_n}t}\cdot M_{X_n}\left(\frac{1}{\sqrt{\lambda_n}}t\right) = e^{-\sqrt{\lambda_n}t}\cdot e^{\lambda_n\left(e^{\frac{1}{\sqrt{\lambda_n}}t}-1\right)}$

$\to$? as $n\to\infty$

Hint: $e^x = e^0 + xe^0 + \frac{x^2}{2}e^0 + \cdots = 1+x+\frac{x^2}{2}+\cdots$

Say $Z\sim N(0,1)$, then Z has MGF $M_Z(t) = e^{\frac{1}{2}t^2}$

So we want to show $-\sqrt{\lambda_n}t + \lambda_n\left(e^{\frac{1}{\sqrt{\lambda_n}}t}-1\right) \xrightarrow{n\to\infty} \frac{1}{2}t^2$

$-\sqrt{\lambda_n} t + \lambda_n \left(\cancel{1} + \frac{1}{\sqrt{\lambda_n}} t + \frac{1}{2}\left(\frac{1}{\sqrt{\lambda_n}} t\right)^2 + \cdots \cancel{-1} \right)$

$= \cancel{-\sqrt{\lambda_n} t} + \cancel{\sqrt{\lambda_n} t} + \frac{1}{2} t^2 + \frac{1}{3!}\frac{1}{\sqrt{\lambda_n}} t^3 + o\left(\frac{1}{\sqrt{\lambda_n}}\right)$

$a_n = o(b_n) \ldots \lim_{n\to\infty} \frac{a_n}{b_n} = 0$

$\frac{1}{n^2} = o\left(\frac{1}{n}\right)$

$\frac{1}{n^2} + \frac{1}{n^3} = o\left(\frac{1}{n}\right)$

Let $\lambda_n \to \infty$

$\to \frac{1}{2} t^2 + \lim_{\lambda_n \to \infty} \left(\frac{1}{3!} \frac{1}{\sqrt{\lambda_n}} t^3 + o\left(\frac{1}{\sqrt{\lambda_n}}\right) \right) = \frac{1}{2} t^2$

So we have shown $\log M_{Z_n}(t) \xrightarrow{n\to\infty} \log M_Z(t) \Leftrightarrow M_{Z_n}(t) \xrightarrow{n\to\infty} M_Z(t)$

By the Continuity Thm, we have $Z_n \xrightarrow{d} Z \sim N(0,1)$.

(Standardized Poisson (λ) r.v. conv. in dist. to $N(0,1)$ as $\lambda \to \infty$). $\square$

Central Limit Theorem (CLT)

Let $X_1, \cdots, X_n$ be iid r.v.s with mean μ, variance $\sigma^2 < \infty$ and MGF $M_X(\cdot)$ defined in a neighborhood of 0.

Let $S_n = \sum_{i=1}^n X_i = X_1 + \cdots + X_n$ (sum) $E[S_n] = n\mu$, $var(S_n) = n\sigma^2$

$\overline{X_n} = \frac{1}{n} S_n = \frac{X_1 + \cdots + X_n}{n}$ (average) $E[\overline{X_n}] = \mu$, $var(\overline{X_n}) = \frac{1}{n^2} var(S_n) = \frac{\sigma^2}{n}$

Standardize S_n : $\frac{S_n - E[S_n]}{\sqrt{var(S_n)}} = \frac{S_n - n\mu}{\sqrt{n}\,\sigma}$

standardize $\overline{X_n}$: $\frac{\overline{X_n} - E[\overline{X_n}]}{\sqrt{var(\overline{X_n})}} = \frac{\overline{X_n} - \mu}{\sigma/\sqrt{n}} = \frac{S_n/n - \mu}{\sigma/\sqrt{n}} = \frac{S_n - n\mu}{\sqrt{n}\,\sigma}$

CLT statement: $\boxed{\frac{S_n - n\mu}{\sqrt{n}\,\sigma} = \frac{\overline{X_n} - \mu}{\sigma/\sqrt{n}} \xrightarrow{d} N(0,1) \text{ as } n \to \infty}$

Pf. Let $Z_n = \frac{S_n - n\mu}{\sqrt{n}\,\sigma} = -\frac{\sqrt{n}\mu}{\sigma} + \frac{1}{\sqrt{n}\sigma} S_n$ $\quad M_{S_n}(t) = (M_X(t))^n$

$\Rightarrow M_{Z_n}(t) = e^{-\frac{\sqrt{n}\mu}{\sigma} t} \cdot M_{S_n}\left(\frac{1}{\sqrt{n}\sigma} \cdot t\right) = e^{-\frac{\sqrt{n}\mu}{\sigma} t} \cdot \left(M_X\left(\frac{1}{\sqrt{n}\sigma} t\right)\right)^n$

Goal: to show $M_{Z_n}(t) \xrightarrow{n\to\infty} e^{\frac{1}{2}t^2}$

$\Leftrightarrow \log M_{Z_n}(t) \xrightarrow{n\to\infty} \frac{1}{2} t^2$

$\log M_{Z_n}(t) = -\frac{\sqrt{n}\mu}{\sigma} t + n \log \underline{M_X\left(\frac{1}{\sqrt{n}\sigma} t\right)}$

$M_X(t) = E[e^{tX}]$
$M_X(0) = 1$

Recall that Taylor expansion:

$M_X\left(\frac{t}{\sqrt{n}\sigma}\right) = M_X(0) + \frac{t}{\sqrt{n}\sigma} \cdot M_X'(0) + \frac{1}{2!}\left(\frac{t}{\sqrt{n}\sigma}\right)^2 \cdot M_X''(0) + \frac{1}{3!}\left(\frac{t}{\sqrt{n}\sigma}\right)^3 \cdot M_X'''(0) + \cdots$

$= 1 + \frac{t}{\sqrt{n}\sigma}\mu + \frac{1}{2!}\left(\frac{t}{\sqrt{n}\sigma}\right)^2 \cdot E[X^2] + \frac{1}{3!}\left(\frac{t}{\sqrt{n}\sigma}\right)^3 \cdot E[X^3] + \cdots$

$$\log M_X\left(\frac{1}{\sqrt{n}\sigma}t\right) = \log\left(1+\boxed{\frac{t}{\sqrt{n}\sigma}\mu + \frac{1}{2}\left(\frac{t}{\sqrt{n}\sigma}\right)^2(\mu^2+\sigma^2) + \frac{1}{3!}\left(\frac{t}{\sqrt{n}\sigma}\right)^3 E[X^3] + \cdots}\right)$$

Recall
$$\log(1+x) = \log(1+0) + x\cdot\frac{d}{dx}\log(1+x)\Big|_{x=0} + \frac{1}{2!}x^2\frac{d^2}{dx^2}\log(1+x)\Big|_{x=0} + \cdots$$
$$= 0 + x\cdot\underbrace{\left(\frac{1}{1+x}\Big|_{x=0}\right)}_{=1} + \frac{1}{2}x^2\cdot\underbrace{\left(-\frac{1}{(1+x)^2}\Big|_{x=0}\right)}_{=-1} + \cdots$$
$$= x - \frac{1}{2}x^2 + \cdots$$

fact: $\log(1+x) \approx x$

$$\boxed{\left(\frac{1}{\sqrt{n}}\right)^3 = o\left(\frac{1}{n}\right)}$$

$$\lim_{n\to\infty}\frac{\left(\frac{1}{\sqrt{n}}\right)^3}{\frac{1}{n}} = \lim\frac{1}{\sqrt{n}} = 0$$

So
$$\log M_X\left(\frac{1}{\sqrt{n}\sigma}t\right) = \boxed{} - \frac{1}{2}\boxed{}^2 + \cdots$$
$$= \boxed{\frac{t}{\sqrt{n}\sigma}\mu + \frac{1}{2}\frac{t^2}{n\sigma^2}(\mu^2+\sigma^2) + o\left(\frac{1}{n}\right)}$$
$$- \frac{1}{2}\left(\frac{t^2}{n\sigma^2}\mu^2 + o\left(\frac{1}{n}\right)\right) + o\left(\frac{1}{n}\right)$$
$$= \frac{t}{\sqrt{n}\sigma}\mu + \underbrace{\frac{1}{2}\frac{t^2}{n\sigma^2}(\mu^2+\sigma^2) - \frac{1}{2}\left(\frac{t^2}{n\sigma^2}\mu^2\right)} + o\left(\frac{1}{n}\right)$$
$$= \frac{t}{\sqrt{n}\sigma}\mu + \frac{1}{2}\frac{t^2}{n\cancel{\sigma^2}}\cancel{\sigma^2} + o\left(\frac{1}{n}\right)$$

So
$$\log M_{Z_n}(t) = -\frac{\sqrt{n}\mu}{\sigma}t + n\log M_X\left(\frac{1}{\sqrt{n}\sigma}t\right)$$
$$= -\frac{\sqrt{n}\mu}{\sigma}t + n\left(\frac{t}{\sqrt{n}\sigma}\mu + \frac{1}{2}\frac{t^2}{n} + o\left(\frac{1}{n}\right)\right)$$
$$= \cancel{-\frac{\sqrt{n}\mu}{\sigma}t + \frac{\sqrt{n}t}{\sigma}\mu} + \frac{1}{2}t^2 + \underbrace{o(1)}_{\to 0}$$

So
$$\lim_{n\to\infty}\log M_{Z_n}(t) = \frac{1}{2}t^2 \iff M_{Z_n}(t) \xrightarrow{n\to\infty} e^{\frac{1}{2}t^2}$$

By the Continuity Theorem, $Z_n \xrightarrow{d} N(0,1)$. $\square$