$g(x,y) \approx g(\mu_X,\mu_Y) + (x-\mu_X)\cdot \frac{\partial g}{\partial x}(\mu_X,\mu_Y) + (y-\mu_Y)\cdot \frac{\partial g}{\partial y}(\mu_X,\mu_Y)$
$+ \frac{1}{2}\Big[(x-\mu_X)^2 \cdot \frac{\partial^2}{\partial x^2} g(\mu_X,\mu_Y) + 2(x-\mu_X)(y-\mu_Y)\frac{\partial^2}{\partial x \partial y} g(\mu_X,\mu_Y) +$
$(y-\mu_Y)^2 \cdot \frac{\partial^2}{\partial y^2} g(\mu_X,\mu_Y)\Big]$

Now: $Z = g(X,Y)$, $E[X]=\mu_X$, $E[Y]=\mu_Y$, $var[X]=\sigma_X^2$, $var[Y]=\sigma_Y^2$
$cov(X,Y)=\sigma_{XY}$

Bivariate Delta method:

$E[Z] \approx$

$cov(X-a, Y-b)$
$= cov(X,Y)$

$E[X-\mu_X]$
$= E[X]-\mu_X = \mu_X - \mu_X = 0$

$var[Z] \approx$

$E[aX] = aE[X]$
$\cancel{E[X]}$
$\underset{(X,Y)}{E}[XY] = \underset{Y}{E}[E[X|Y]]$

$Z = g(X,Y) \approx g(\mu_X,\mu_Y) + (X-\mu_X)\cdot\frac{\partial g}{\partial x}(\mu_X,\mu_Y) + (Y-\mu_Y)\cdot \frac{\partial g}{\partial y}(\mu_X,\mu_Y)$
$+\frac{1}{2}\Big[(X-\mu_X)^2\cdot\frac{\partial^2}{\partial x^2}g(\mu_X,\mu_Y) + 2(X-\mu_X)(Y-\mu_Y)\cdot\frac{\partial^2}{\partial x\partial y}g(\mu_X,\mu_Y)$
$+(Y-\mu_Y)^2\cdot\frac{\partial^2}{\partial y^2}g(\mu_X,\mu_Y)\Big]$

$E[Z] \approx g(\mu_X,\mu_Y) + \underset{=0}{E[X-\mu_X]}\cdot\frac{\partial g}{\partial x}(\mu_X,\mu_Y) + 0$
$+\frac{1}{2}\Big[\underbrace{E[(X-\mu_X)^2]}_{=\sigma_X^2}\cdot\frac{\partial^2}{\partial x^2}g(\mu_X,\mu_Y) + 2\underbrace{E[(X-\mu_X)(Y-\mu_Y)]}_{=\sigma_{XY}}\frac{\partial^2}{\partial x\partial y}g(\mu_X,\mu_Y)$
$+\underbrace{E[(Y-\mu_Y)^2]}_{=\sigma_Y^2}\cdot\frac{\partial^2}{\partial y^2}g(\mu_X,\mu_Y)\Big]$

$var[Z] \approx var\Big[\cancel{g(\mu_X,\mu_Y)} + (X-\mu_X)\cdot\frac{\partial g}{\partial x}(\mu_X,\mu_Y) + (Y-\mu_Y)\cdot\frac{\partial g}{\partial y}(\mu_X,\mu_Y)\Big]$
$= \left(\frac{\partial g}{\partial x}(\mu_X,\mu_Y)\right)^2\cdot var(X) + 2\left(\frac{\partial g}{\partial x}(\mu_X,\mu_Y)\right)\cdot\left(\frac{\partial g}{\partial y}(\mu_X,\mu_Y)\right)\cdot cov(X,Y) + \left(\frac{\partial g}{\partial y}(\mu_X,\mu_Y)\right)^2 var(Y)$

$var[aX+bY] = a\,var[X] + b\,var[Y]$ ✗
$= a^2 var[X] + b^2 var[Y]$ ✗
$\| $
$\underline{a^2 var(X) + 2ab\,cov(X,Y) + b^2 var(Y)}$

$var[aX] = a^2 var[X]$
$\| $
$E[(aX - aE[X])^2]$
$= a^2 E[(X-E[X])^2]$
$= a^2 var[X]$

$var[X+Y] = E[(X+Y-\mu_X-\mu_Y)^2] = E[((X-\mu_X)+(Y-\mu_Y))^2]$
$= E[(X-\mu_X)^2 + 2(X-\mu_X)(Y-\mu_Y) + (Y-\mu_Y)^2]$
$= E[(X-\mu_X)^2] + 2E[(X-\mu_X)(Y-\mu_Y)] + E[(Y-\mu_Y)^2]$

$var[X+Y] = var[X] + 2cov(X,Y) + var[Y]$

Continuity Theorem (conv of MGF $\Rightarrow$ conv of CDF)

Let $X_1, X_2, \cdots$ be a sequence of R.V.s with CDFs $F_1(\cdot), F_2(\cdot), \ldots$ and $M_{X_1}(\cdot), M_{X_2}(\cdot), \cdots$. Suppose that

$$M_{X_n}(t) \to M_X(t) \text{ as } n \to \infty \text{ for all } t \in \mathbb{R}$$

where X is another R.V. with CDF $F(\cdot)$ and MGF $M_X(\cdot)$.

Then $F_n(x) \to F(x)$ as $n \to \infty$ for all x at which F is continuous

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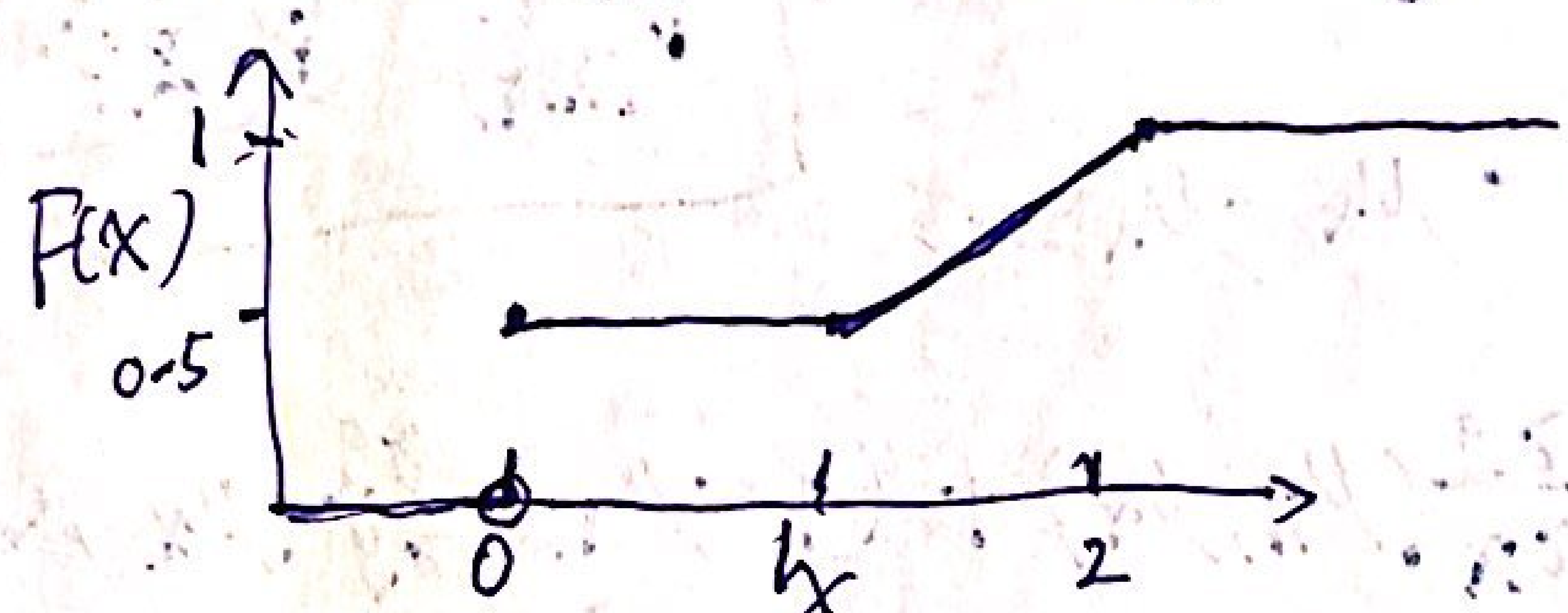
e.g. $X = 0$ with probability 0.5, & $X \sim \text{Unif}[1,2]$ with probability 0.5

Recall $F(x) = \mathbb{P}(X \le x) = \begin{cases} 0 & \text{if } x < 0 \\ 0.5 & x = 0 \\ 0.5 & 0 < x \le 1 \\ 0.5 + 0.5(x-1) & 1 < x \le 2 \\ 1 & x > 2 \end{cases}$

$\mathbb{P}(X \le 0.5) = \mathbb{P}(X = 0)$

$\mathbb{P}(X \le x) = \mathbb{P}(X = 0) + \mathbb{P}(1 < X \le x) = 0.5 + 0.5 \cdot (x-1)$



F is cont. everywhere except at 0.

Example: Poisson(λ) converges to a normal dist as $\lambda \to \infty$.

Pf: Let $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$ be an increasing sequence $\lambda_n \to \infty$ as $n \to \infty$

Let $X_n \sim \text{Poisson}(\lambda_n)$. $E[X_n] = \lambda_n$; $\text{var}[X_n] = \lambda_n$

Recall X_n has MGF $M_{X_n}(t) = e^{\lambda_n(e^t - 1)}$ $t \in \mathbb{R}$

Standardize X_n as

$$Z_n = \frac{X_n - \lambda_n}{\sqrt{\lambda_n}} = -\sqrt{\lambda_n} + \left(\frac{1}{\sqrt{\lambda_n}}\right) X_n$$

Try to show that Z_n converges to a $N(0,1)$ R.V. as $\lambda_n \to \infty$ $(n \to \infty)$

So $M_{Z_n}(t) = e^{-\sqrt{\lambda_n} t} \cdot M_{X_n}\left(\frac{1}{\sqrt{\lambda_n}} t\right) = e^{-\sqrt{\lambda_n} t} \cdot e^{\lambda_n (e^{\frac{1}{\sqrt{\lambda_n}} t} - 1)}$

$\to ?$ as $n \to \infty$

Say $Z \sim N(0,1)$, then Z has MGF $M_Z(t) = e^{\frac{1}{2}t^2}$

So we want to show $-\sqrt{\lambda_n} t + \lambda_n \left(e^{\frac{1}{\sqrt{\lambda_n}} t} - 1\right) \xrightarrow{n \to \infty} \frac{1}{2} t^2$

Hint: $e^x = e^0 + x e^0 + \frac{x^2}{2} e^0 + \cdots = 1 + x + \frac{x^2}{2} + \cdots$