which is the MGF of ...

so $X+Y \sim N(\mu_1+\mu_2,\ \sigma_1^2+\sigma_2^2)$

$X_1, \cdots, X_n$ are iid R.V.s. What is the MGF of $\sum_{i=1}^{n} X_i$?

$$M_{\sum_{i=1}^{n} X_i}(t) \overset{\text{indep}}{=} M_{X_1}(t) \cdot M_{X_2}(t) \cdots M_{X_n}(t)$$

$$\overset{\text{ident. dist}: M_{X_1}(t) = M_{X_2}(t) = \cdots = M_{X_n}(t) \triangleq M_X(t)}{=} \left(M_X(t)\right)^n$$

Example: $X_1, \cdots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$

Recall $M_X(t) = e^{\mu t} \cdot e^{\sigma^2 t^2/2}$

$$M_{\sum_{i=1}^{n} X_i}(t) = \left(e^{\mu t} e^{\sigma^2 t^2/2}\right)^n = e^{n\mu t} \cdot e^{n\sigma^2 t^2/2}$$

so $\sum_{i=1}^{n} X_i \sim N(n\mu,\ n\sigma^2)$

Joint MGF: a pair of R.Vs. (X, Y); $X, Y \in \mathbb{R}$

If both (X, Y) are discrete:

$$M_{X,Y}(s,t) = E\left[e^{sX+tY}\right] = E\left[e^{(s,t)^T \cdot \binom{X}{Y}}\right] \qquad s,t \in \mathbb{R}$$

$$= \sum_x \sum_y e^{sx+ty}\, \mathbb{P}(X=x, Y=y)$$

or if X & Y are cont.

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{sx+ty} f(x,y)\, dx\, dy$$

where $f(x,y)$ is the joint pdf of (X,Y)

If X and Y are independent,

$$M_{X,Y}(s,t) = E\left[e^{sX+tY}\right] = E\left[e^{sX} \cdot e^{tY}\right]$$

$$\overset{\text{indep}}{=} E\left[e^{sX}\right] \cdot E\left[e^{tY}\right] = M_X(s) \cdot M_Y(t).$$

To obtain joint moments of (X, Y)

$$E\left[X^r Y^v\right] = \cancel{\frac{\partial^{r+v} M_{X,Y}}{\partial^r s\, \partial^v t}} \ \frac{\partial^{r+v} M_{X,Y}(s,t)}{\partial s^r\, \partial t^v}\Bigg|_{s=0,\, t=0}$$

(r,v)-th moment of (X,Y)

Properties:
① $M_{X,Y}(s,0) = E\left[e^{sX}\right] = M_X(s)$
② $M_{X,Y}(0,t) = E\left[e^{tY}\right] = M_Y(t)$
③ $M_{X,Y}(t,t) = E\left[e^{t(X+Y)}\right] = M_{X+Y}(t)$

Delta method (approximation)

Q: Suppose $E[X] = \mu_X$, $var[X] = \sigma_X^2$, and $Y = g(X)$

How to find $E[Y]$ and $var[Y]$?

Ideal answer: use the definition

$$E[Y] = E[g(X)] = \begin{cases} \sum_x g(x) \cdot \mathbb{P}(X = x) & \text{if } X \text{ discrete} \\ \int_{-\infty}^{\infty} g(x) f(x)\, dx & f \text{ is the pdf of } X \text{ cont} \end{cases}$$

But $\mathbb{P}(X=x)$ ~~and~~ or $f(x)$ is unknown.

The dist of X is only known up to $E[X]$ and $Var[X]$.

Approximate answer (by Delta method):

By Taylor Expansion: (approx. $g(X)$ by a polynomial function of X)

$$Y = g(X) = g(\mu_X) + (X-\mu_X) \cdot g'(\mu_X) + \frac{(X-\mu_X)^2}{2} \cdot g''(\mu_X) + \cdots$$

$$E[Y] = g(\mu_X) + g'(\mu_X) \cdot E[X-\mu_X] + \frac{g''(\mu_X)}{2} E[(X-\mu_X)^2] + \cdots$$

$$\approx g(\mu_X) + g'(\mu_X) \cdot \underbrace{(E[X]-\mu_X)}_{=0} + \frac{g''(\mu_X)}{2} \sigma_X^2$$

c constant ; X r.v.

$E[cX] = cE[X]$

$E[X+c] = E[X] + c$

$var[cX] = c^2 Var[X]$

$var[X+c] = var[X]$

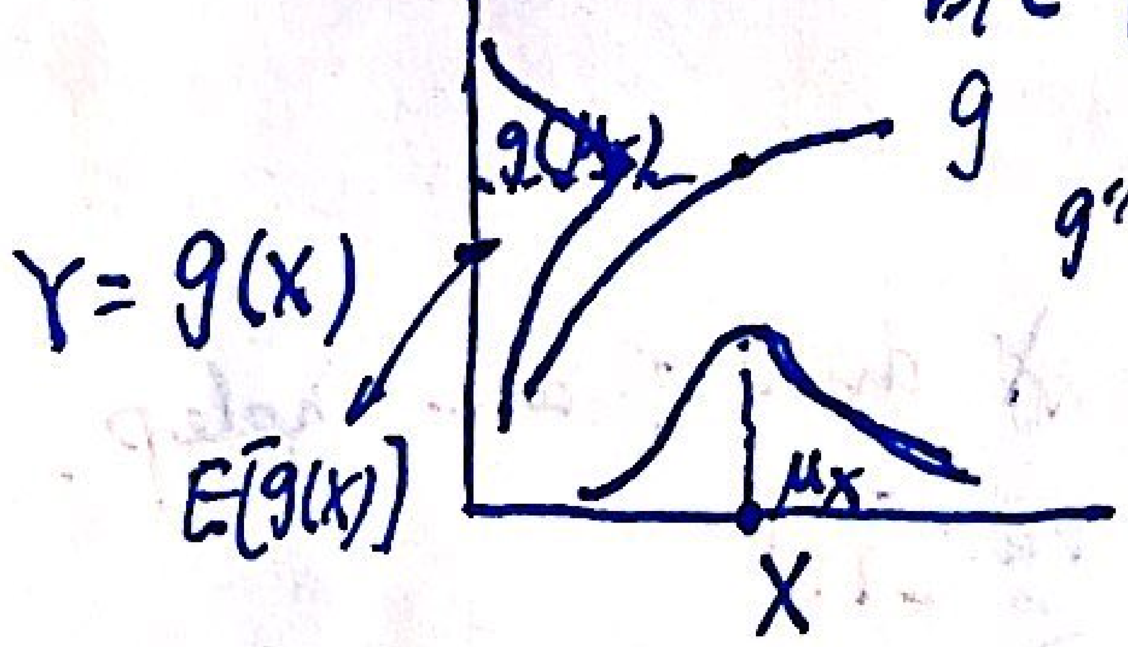
So $E[Y] = E[g(X)] \approx g(\mu_X) + \frac{g''(\mu_X)}{2} \sigma_X^2$

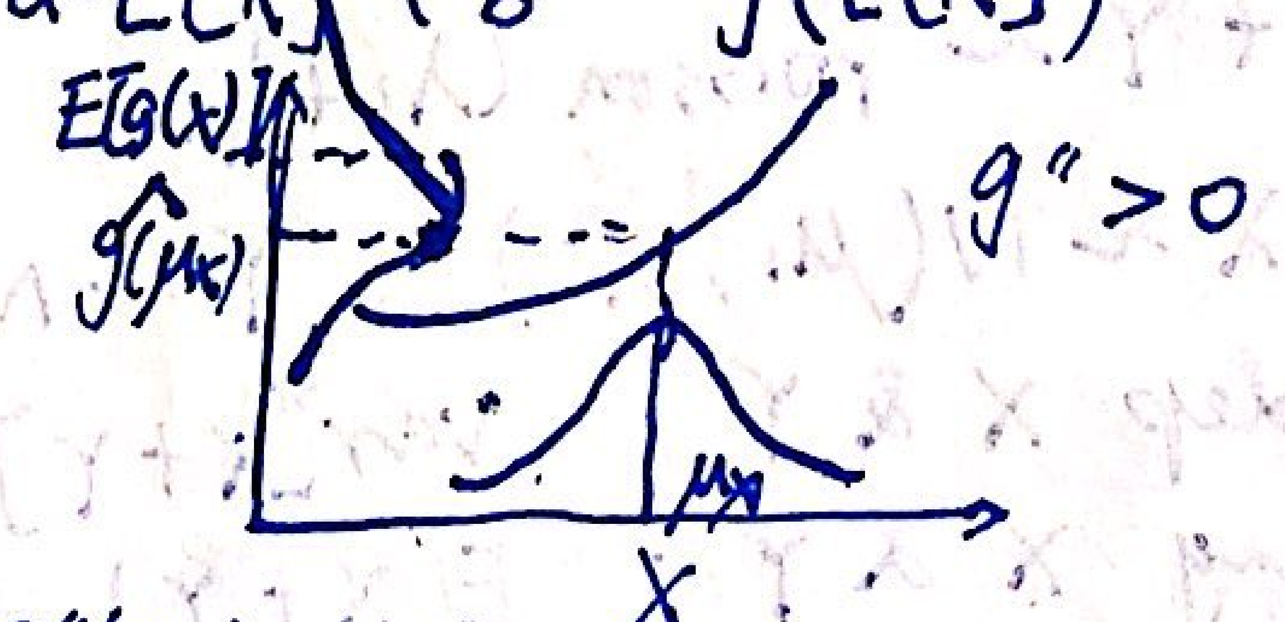
special case: $g(X) = aX + b$

then $g''(x) = 0 \quad \forall x \in \mathbb{R}$

$E[g(X)] = g(\mu_X) = g(E[X])$

b/c $E[aX+b] = a \cdot E[X] + b = g(E[X])$





$$g(\mu_X) > E[g(X)] \approx g(\mu_X) + \underbrace{\frac{g''(\mu_X)}{2} \sigma_X^2}_{<0}$$

If we only know $E[X] = \mu_X$, and $Y = g(X)$, then

$E[Y] \approx g(\mu_X)$ is the best we can do.

How about var[Y]?

$$\mathrm{var}[Y] \approx (g'(\mu_X))^2 \cdot \mathrm{var}(X-\mu_X)$$
$$= (g'(\mu_X))^2 \cdot \mathrm{var}(X) = (g'(\mu_X))^2 \cdot \sigma_X^2 .$$

To summarize, Delta method gives:

$$E[g(X)] \approx g(\mu_X) + \frac{g''(\mu_X)}{2}\sigma_X^2$$
$$\mathrm{var}[g(X)] \approx (g'(\mu_X))^2 \cdot \sigma_X^2$$

Example: $X \sim \text{Uniform}[10, 20]$. Find the exact and approximate mean/expectation and variance of $Y = \frac{1}{X}$.

Answer: Exact answers:

$$E[Y] = E\left[\frac{1}{X}\right] = \int_{10}^{20} \frac{1}{x} \cdot \frac{1}{20-10}\, dx = \frac{1}{10}\int_{10}^{20} \frac{1}{x}\, dx$$
$$= \frac{1}{10} \log x \Big|_{10}^{20} = \frac{1}{10}(\log 20 - \log 10) = \frac{1}{10}\cdot \log 2 \approx 0.0693$$

$$E[Y^2] = E\left[\frac{1}{X^2}\right] = \int_{10}^{20} \frac{1}{x^2} \cdot \frac{1}{20-10}\, dx = \frac{1}{10}\int_{10}^{20} \frac{1}{x^2}\, dx$$
$$= \frac{1}{10}\left(-\frac{1}{x}\right)\Big|_{10}^{20} = \frac{1}{10}\left(-\frac{1}{20} + \frac{1}{10}\right) = \frac{1}{200}$$

$$\mathrm{var}[Y] = E[Y^2] - (E[Y])^2 = \frac{1}{200} - \left(\frac{1}{10}\log 2\right)^2 \approx 0.000195$$

Approximate answers: (by Delta method)

$$E[Y] \approx g(\mu_X) + \frac{g''(\mu_X)}{2}\sigma_X^2 \Leftarrow$$
$$\mathrm{var}[Y] \approx (g'(\mu_X))^2 \cdot \sigma_X^2$$

$X \sim \text{Uniform}[10, 20] \Rightarrow \mu_X = 15,\ \sigma_X^2 = \frac{25}{3}$

$g(x) = \frac{1}{x},\ g'(x) = -\frac{1}{x^2}$

$g''(x) = \frac{2}{x^3}$

then $$E[Y] \approx \frac{1}{15} + \frac{2}{(15)^3}\cdot\frac{1}{2}\cdot\frac{25}{3} = 0.0691$$

$$\mathrm{var}[Y] \approx \left(-\frac{1}{15^2}\right)^2 \cdot \frac{25}{3} = 0.000165$$

Bivariate Taylor expansion: $g(x, y)$ expand it at (μ_X, μ_Y)

$$g(x,y) = g(\mu_X, \mu_Y) + (x-\mu_X, y-\mu_Y)^T \cdot \begin{pmatrix} \frac{\partial g}{\partial x}(\mu_X, \mu_Y) \\ \frac{\partial g}{\partial y}(\mu_X, \mu_Y) \end{pmatrix}$$
$$+ \frac{1}{2}(x-\mu_X, y-\mu_Y)^T \begin{bmatrix} \frac{\partial^2}{\partial x^2} g(\mu_X, \mu_Y) & \frac{\partial^2}{\partial x \partial y} g(\mu_X, \mu_Y) \\ \frac{\partial^2}{\partial y \partial x} g(\mu_X, \mu_Y) & \frac{\partial^2}{\partial y^2} g(\mu_X, \mu_Y) \end{bmatrix} \begin{pmatrix} x-\mu_X \\ y-\mu_Y \end{pmatrix} \cancel{(x-\mu_X, y-\mu_Y)}$$
$$+ \cdots$$

$$g(x,y) \approx g(\mu_X,\mu_Y) + (x-\mu_X)\cdot\frac{\partial g}{\partial x}(\mu_X,\mu_Y) + (y-\mu_Y)\cdot\frac{\partial g}{\partial y}(\mu_X,\mu_Y)$$
$$+ \frac{1}{2}\left[(x-\mu_X)^2\cdot\frac{\partial^2}{\partial x^2}g(\mu_X,\mu_Y) + 2(x-\mu_X)(y-\mu_Y)\frac{\partial^2}{\partial x\partial y}g(\mu_X,\mu_Y) + (y-\mu_Y)^2\cdot\frac{\partial^2}{\partial y^2}g(\mu_X,\mu_Y)\right]$$

Now: $Z = g(X,Y)$, $E[X]=\mu_X$, $E[Y]=\mu_Y$, $var[X]=\sigma_X^2$, $var[Y]=\sigma_Y^2$, $cov(X,Y)=\sigma_{XY}$

Bivariate Delta method:

$E[Z] \approx$

$var[Z] \approx$