Know $E[X] = np$; $var[X] = np(1-p)$

$$E[X^2] = (E[X])^2 + var[X] = n^2p^2 + np(1-p) = n^2p^2 + np - np^2$$

verify $E[X]$, $E[X^2]$ by $M_X(t)$:

$$M_X'(t) = n(e^tp+1-p)^{n-1} \cdot e^tp \Rightarrow M_X'(0) = n \cdot p.$$

$$M_X''(t) = n(n-1)(e^tp+1-p)^{n-2} \cdot e^tp \cdot e^tp + n(e^tp+1-p)^{n-1} \cdot e^tp.$$

$$\Rightarrow M_X''(0) = n(n-1)p^2 + np = n^2p^2 - np^2 + np.$$

② $X \sim$ Poisson(λ) $\lambda > 0$

$$P(X=x) = \frac{\lambda^x \cdot e^{-\lambda}}{x!}, \quad x = 0, 1, 2, \ldots \qquad \sum_{x=0}^{\infty} P(X=x) = 1$$

$$= \sum_{x=0}^{\infty} \frac{\lambda^x e^{-\lambda}}{x!} = 1 \quad \forall \lambda > 0$$

$$M_X(t) = \sum_{x=0}^{\infty} e^{tx} \cdot P(X=x) = \sum_{x=0}^{\infty} e^{tx} \cdot \frac{\lambda^x \cdot e^{-\lambda}}{x!} = \sum_{x=0}^{\infty} \frac{(e^t \cdot \lambda)^x e^{-\lambda}}{x!}$$

$$= \underbrace{\boxed{\sum_{x=0}^{\infty} \frac{(e^t\lambda)^x \cdot e^{-e^t \cdot \lambda}}{x!}}}_{=1} \cdot e^{e^t\lambda} \cdot e^{-\lambda} = e^{e^t\lambda} \cdot e^{-\lambda}$$

$$= e^{\lambda(e^t-1)}$$

Know $E[X] = \lambda$, $var[X] = \lambda \Rightarrow E[X^2] = (E[X])^2 + var[X] = \lambda^2 + \lambda$

$$M_X'(t) = e^{\lambda(e^t-1)} \cdot \lambda e^t \Rightarrow M_X'(0) = \lambda$$

$$M_X''(t) = e^{\lambda(e^t-1)} \cdot \lambda e^t \cdot \lambda e^t + e^{\lambda(e^t-1)} \cdot \lambda e^t$$

$$\Rightarrow M_X''(0) = \lambda^2 + \lambda$$

③ $X \sim Exp(\lambda)$ $\lambda > 0$

pdf : $f(x) = \begin{cases} \lambda \cdot e^{-\lambda x} & x > 0 \\ 0 & x \leq 0 \end{cases}$ $\quad \boxed{\int_0^{\infty} \lambda e^{-\lambda x} dx = 1 \quad \forall \lambda > 0}$

$$M_X(t) = E[e^{tX}] = \int_0^{\infty} e^{tx} \cdot f(x)\,dx = \int_0^{\infty} e^{tx} \cdot \lambda \cdot e^{-\lambda x} dx$$

$$= \int_0^{\infty} \lambda e^{-(\lambda - t)x} dx \overset{\text{if } \lambda - t > 0}{=\!=} \int_0^{\infty} (\lambda - t) e^{-(\lambda-t)x} \cdot \frac{\lambda}{\lambda - t} dx$$

$$= \frac{\lambda}{\lambda - t} \quad \text{if } t < \lambda.$$ Check: $M_X'(0) = E[X]$, $M_X''(0) = E[X^2]$

$X \sim N(0,1)$ pdf: $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$, $x \in \mathbb{R}$

$$\boxed{\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = 1}$$

$$M_X(t) = \int_{-\infty}^{+\infty} e^{tx} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-(\frac{x^2}{2} - tx)} dx$$

$$= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-(\frac{(x-t)^2}{2} - \frac{t^2}{2})} dx$$

$$= e^{t^2/2} \underbrace{\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-t)^2}{2}} \underbrace{dx}_{=d(x-t)}}_{=1}$$

$$= e^{t^2/2}$$

Q: how to make $\frac{x^2}{2} - tx = \frac{(x-\bigcirc)^2}{2} + \bigcirc$

$$\frac{x^2 - 2tx + t^2 - t^2}{2} = \frac{(x-t)^2 - t^2}{2}$$

$(x-t)^2 = x^2 - 2tx + t^2$

$dx = d(x-t)$ + const

$dx^2 = 2dx$

$$\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-t)^2}{2}} dx \overset{y=x-t}{=\!=} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy = 1$$

Properties of MGF:

(1) If X has MGF $M_X(t)$ and $Y = a + bX$, $a, b \in \mathbb{R}$

Then Y has MGF $M_Y(t) = e^{at} \cdot M_X(bt)$

Pf: $M_Y(t) = E[e^{tY}] = E[e^{t(a+bX)}] = E[e^{at} \cdot e^{btX}]$

$$= e^{at} \cdot \underbrace{E[e^{(bt)X}]}_{=M_X(bt)} = e^{at} \cdot M_X(bt)$$

Example: $Y \sim N(\mu, \sigma^2)$ pdf: $f(y) = \frac{1}{\sqrt{2\pi}\sigma} \cdot e^{-\frac{(y-\mu)^2}{2}}$

Note that we can write $Y = \mu + \sigma X$, where $X \sim N(0,1)$

$E[X] = 0$, $var[X] = 1$, $E[Y] = \mu + \sigma E[X] = \mu$, $var[Y] = \sigma^2 var[X] = \sigma^2$

Recall $M_X(t) = e^{t^2/2}$. Then

$M_Y(t) = e^{\mu t} \cdot M_X(\sigma t) = e^{\mu t} \cdot e^{\frac{\sigma^2 t^2}{2}}$

(2) X and Y are independent R.V.s w/ MGFs $M_X(t)$ and $M_Y(t)$ respectively. Then the MGF of $X+Y$ is

$$M_{X+Y}(t) = M_X(t)\, M_Y(t)$$

Recall: independence : $\mathbb{P}(X\in\mathcal{X}, Y\in\mathcal{Y}) = \mathbb{P}(X\in\mathcal{X})\cdot\mathbb{P}(Y\in\mathcal{Y})$

$\mathbb{P}(X\in\mathcal{X} \mid Y\in\mathcal{Y}) = \mathbb{P}(X\in\mathcal{X})$

$E[g(X)\cdot h(Y)] = E[g(X)]\cdot E[h(Y)]$

Pf: $M_{X+Y}(t) = E[e^{t(X+Y)}] = E[e^{tX}\cdot e^{tY}]$

$\overset{\text{indep}}{=} E[e^{tX}]\cdot E[e^{tY}] = M_X(t)\cdot M_Y(t)$

Example 1: $X \sim \text{Binomial}(n_1, p)$, $Y\sim \text{Binomial}(n_2, p)$

X and Y are independent.

Recall $M_X(t) = (e^t p + 1 - p)^{n_1}$, $M_Y(t) = (e^t p + 1 - p)^{n_2}$

then $M_{X+Y}(t) = M_X(t)M_Y(t) = (e^t p + 1 - p)^{n_1+n_2}$

which is the MGF of a binomial r.v.

so $X+Y \sim \text{Binomial}(n_1+n_2, p)$

Example 2: $X\sim \text{Poisson}(\lambda_1)$, $Y \sim \text{Poisson}(\lambda_2)$, X and Y are indep.

Recall $M_X(t) = e^{\lambda_1(e^t-1)}$, $M_Y(t) = e^{\lambda_2(e^t-1)}$

then $M_{X+Y}(t) = M_X(t)M_Y(t) = e^{\lambda_1(e^t-1)}\cdot e^{\lambda_2(e^t-1)} = e^{(\lambda_1+\lambda_2)(e^t-1)}$

which is the MGF of a Poisson r.v.

so $X+Y\sim \text{Poisson}(\lambda_1+\lambda_2)$

Example 3: $X\sim N(\mu_1, \sigma_1^2)$, $Y\sim N(\mu_2,\sigma_2^2)$, X and Y are indep.

For indep X & Y: $\text{var}[X+Y] = \text{var}[X] + \text{var}[Y]$

For any X & Y: $E[X+Y] = E[X] + E[Y]$

so $E[X+Y] = \mu_1+\mu_2$, $\text{var}[X+Y] = \sigma_1^2+\sigma_2^2$

Q: $X+Y \sim N(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$?

Recall: $M_X(t) = e^{\mu_1 t} e^{\sigma_1^2 t^2/2}$, $M_Y(t) = e^{\mu_2 t} e^{\sigma_2^2 t^2/2}$

then $M_{X+Y}(t) = M_X(t)M_Y(t) = e^{(\mu_1+\mu_2)t}\, e^{(\sigma_1^2+\sigma_2^2)t^2/2}$

which is the MGY of a Normal r.v.

So $X+Y \sim N(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$

$X_1, \cdots, X_n$ are iid R.V.s . What is the MGF of $\sum_{i=1}^{n} X_i$?

$$M_{\sum_{i=1}^{n} X_i}(t) \overset{\text{indep}}{=} M_{X_1}(t) \cdot M_{X_2}(t) \cdots M_{X_n}(t)$$

$$\underset{=}{\text{ident. dist}}: M_{X_1}(t) = M_{X_2}(t) = \cdots = M_{X_n}(t) \overset{d}{=} M_X(t)$$

$$= \left(M_X(t)\right)^n$$

Example: $X_1, \cdots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$

Recall $M_X(t) = e^{\mu t} \cdot e^{\sigma^2 t^2/2}$

$$M_{\sum_{i=1}^{n} X_i}(t) = \left(e^{\mu t} e^{\sigma^2 t^2/2}\right)^n = e^{n\mu t} \cdot e^{n\sigma^2 t^2/2}$$

So $\sum_{i=1}^{n} X_i \sim N(n\mu, n\sigma^2)$