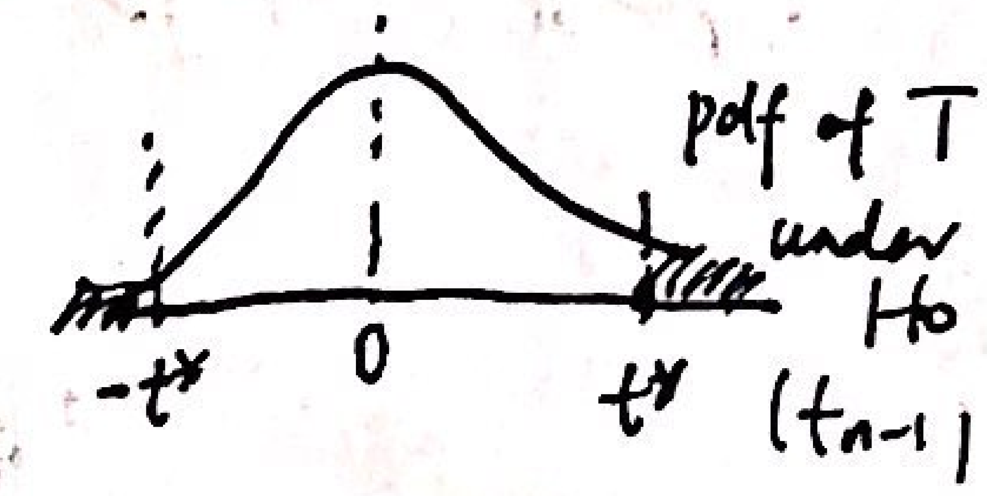
Define $T = \dfrac{\bar{X}_n - \mu_0}{S_n/\sqrt{n}}$ t statistic

Realization of T on $x_1, \dots, x_n$: t^* (e.g. $t^* = 3.71$)

Decision : approach 1: p-value approach : $\left.\begin{array}{l}\text{dist of } T \text{ under } H_0 \\ \text{realization } t^*\end{array}\right\} \Rightarrow$ p-value

p-value is a transformation of t^*

$H_1: \mu \neq \mu_0$: $\triangleq \mathbb{P}_{H_0}(|T| \geq |t^*|)$



p-value : the prob. that T takes values 'more extreme' (determined by H_1) than t^* under H_0

or

$H_1: \mu > \mu_0$: $\triangleq \mathbb{P}_{H_0}(T \geq t^*)$

or

$H_1: \mu < \mu_0$: $\triangleq \mathbb{P}_{H_0}(T \leq t^*)$

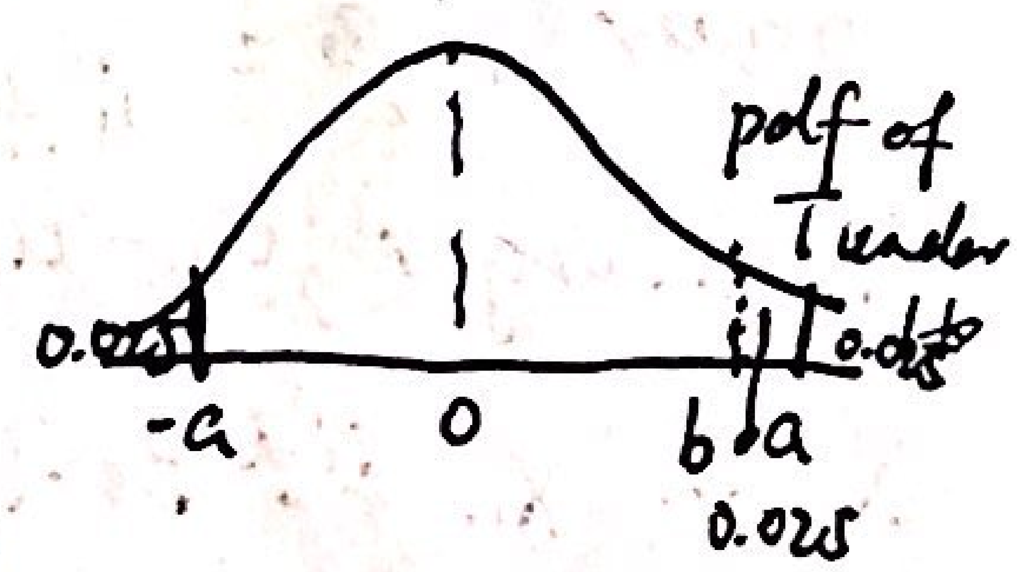
Decision: if p-value $\leq \alpha \Rightarrow$ reject H_0

approach 2: rejection region based on α and the dist of T under H_0

e.g. $\alpha = 0.05$

$H_1: \mu \neq \mu_0$: rejection region

$\{T \leq -a\} \cup \{T \geq a\}$



or $H_1: \mu > \mu_0$: $\{T \geq b\}$

or $H_1: \mu < \mu_0$: $\{T \leq -b\}$

Decision: if $t^* \in$ rejection region $\Rightarrow$ reject H_0

Type I error $= \mathbb{P}_{H_0}(\text{test rejects } H_0) = \mathbb{P}_{H_0}(T \in \text{rejection region})$

ideal case: rejection region is defined to have prob $= \alpha$ under H_0

$\stackrel{\Rightarrow}{=} \alpha$ (significance level)

Type II error $= \mathbb{P}_{H_1}(\text{test does not reject } H_0) = \mathbb{P}_{H_0}(T \notin \text{rejection region})$

$\triangleq \beta$.

power $\triangleq 1-\beta = \mathbb{P}_{H_1}(\text{test rejects } H_0) = \mathbb{P}_{H_1}(T \in \text{rejection region})$

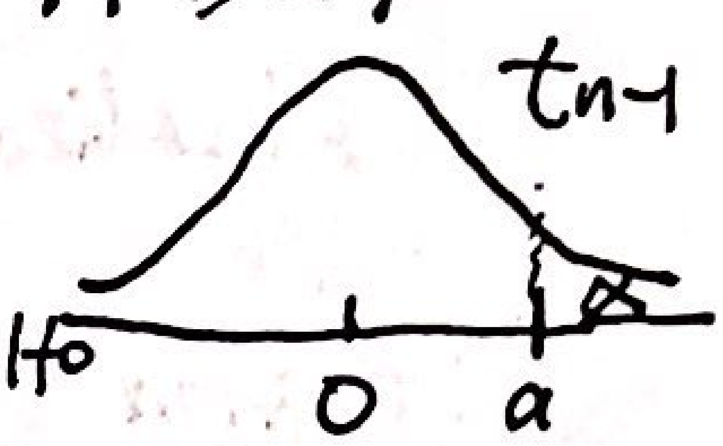
$\searrow$ computatable when alternative hypothesis H_1 is simple, e.g. $H_1: \mu = \mu_1 \neq \mu_0$

under H_0: $\underset{\overset{\|}{T}}{\dfrac{\bar{X}_n - \mu_0}{S_n/\sqrt{n}}} \sim t_{n-1}$ vs. under H_1: $\underset{\overset{\|}{T + \frac{\mu_0 - \mu_1}{S_n/\sqrt{n}}}}{\dfrac{\bar{X}_n - \mu_1}{S_n/\sqrt{n}}} \sim t_{n-1}$

WLOG, assume $\mu_1 > \mu_0$, the rejection region should take the form $\{T \geq a\}$

Given the sig. level α, find a s.t. $\mathbb{P}_{H_0}(T \geq a) = \alpha$

a is the $(1-\alpha)$ quantile of t_{n-1}.



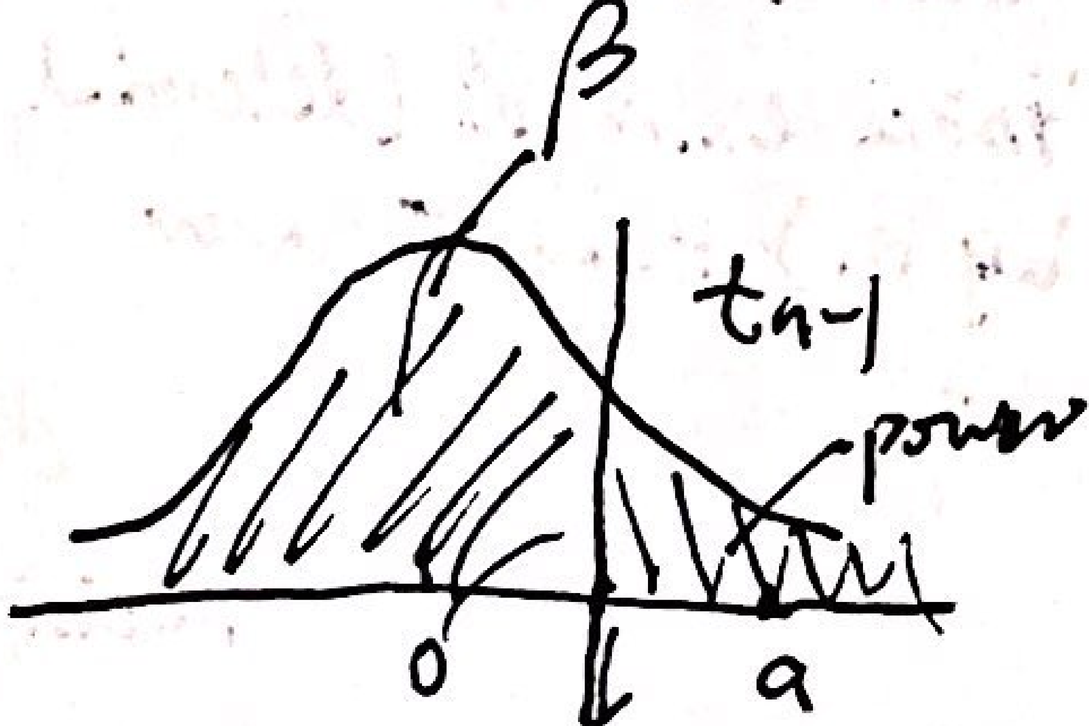
test: $T = \dfrac{\bar{X}_n - \mu_0}{S_n/\sqrt{n}} \overset{\text{realization}}{\Longrightarrow} t^* = \dfrac{\bar{x}_n - \mu_0}{s_n/\sqrt{n}} \Rightarrow \begin{cases} \text{if } t^* \geq a, \text{ reject } H_0 \\ \text{o.w.; do not reject } H_0 \end{cases}$

Type I error $= \alpha$

Type II error $\beta = \mathbb{P}_{H_1}(T < a) = \mathbb{P}_{H_1}\Big(\underbrace{\dfrac{\bar{X}_n - \mu_0}{S_n/\sqrt{n}}}_{\not\sim t_{n-1} \text{ under } H_1} < a \Big)$

$= \mathbb{P}_{H_1}\Big(\underbrace{\dfrac{\bar{X}_n - \mu_1}{S_n/\sqrt{n}}}_{\sim t_{n-1} \text{ under } H_1} + \dfrac{\mu_1 - \mu_0}{S_n/\sqrt{n}} < a \Big)$

$= \mathbb{P}_{H_1}\Big(\dfrac{\bar{X}_n - \mu_1}{S_n/\sqrt{n}} < a - \dfrac{\mu_1 - \mu_0}{S_n/\sqrt{n}} \Big)$



S_n is treated as fixed

$= F\Big(a - \dfrac{\mu_1 - \mu_0}{S_n/\sqrt{n}}\Big)$ where F is the CDF of t_{n-1} $\quad a - \dfrac{\mu_1 - \mu_0}{S_n/\sqrt{n}}$

power $1 - \beta = \mathbb{P}_{H_1}(T \geq a) = \backslash\backslash\backslash\backslash$

The larger $\mu_1 - \mu_0$ is, the greater the power.

$X_1, \cdots, X_n \overset{iid}{\sim} Ber(p)$ $\quad E[X_i] = p = \mathbb{P}(X_i = 1)$ $\quad var[X_i] = p(1-p)$

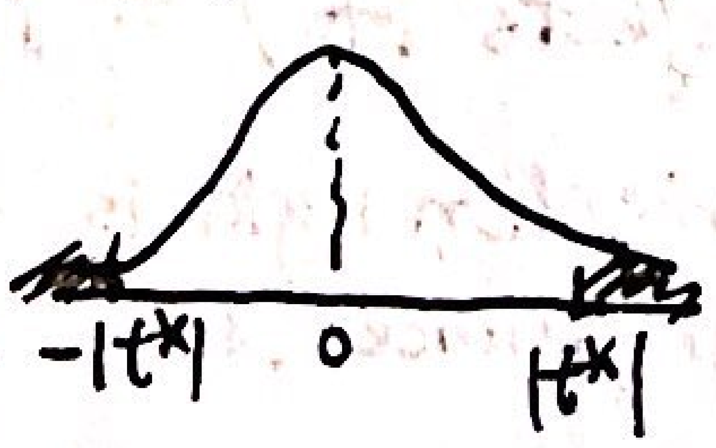
$H_0: p = p_0$ (e.g. $\frac{1}{2}$) $\quad H_1: p \neq p_0$ ($>$, $<$)

$\hat{p} = \bar{X}_n$

under H_0: by CLT $\hat{p} \overset{approx}{\sim} N\Big(p_0, \dfrac{p_0(1-p_0)}{n}\Big)$

$T = \dfrac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} \overset{approx}{\sim} N(0,1)$ $\quad$ realization t^*

p-value $= \mathbb{P}_{H_0}(|T| \geq |t^*|) =$



$= \begin{cases} \text{if } \leq \alpha, \text{ reject } H_0 \\ \text{o.w. do not reject } H_0 \end{cases}$

↑

$H_1: p \neq p_0$

$X_1, \cdots, X_n \overset{iid}{\sim} N(\mu_X, \sigma_X^2)$, $Y_1, \cdots, Y_m \overset{iid}{\sim} N(\mu_Y, \sigma_Y^2)$

$X_1, \cdots, X_n, Y_1, \cdots, Y_m$ independent

$H_0: \mu_X = \mu_Y$ $\quad H_1: \mu_X \neq \mu_Y$ ($<$, $>$)

$\Updownarrow$

$\mu_X - \mu_Y = 0$

Test statistic: $\hat{\mu}_X - \hat{\mu}_Y = \bar{X}_n - \bar{Y}_m \sim N\left(\mu_X - \mu_Y, \frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}\right)$

under H_0: $\bar{X}_n - \bar{Y}_m \sim N\left(0, \frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}\right)$

$$T = \frac{\bar{X}_n - \bar{Y}_m}{\sqrt{\frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}}} \sim N(0,1)$$ assuming σ_X^2 and σ_Y^2 are known

realization t^*

$$\text{p-value} = \begin{cases} \mathbb{P}_{H_0}(|T| \geq |t^*|) & \text{if } H_1: \mu_X \neq \mu_Y \\ \mathbb{P}_{H_0}(T \geq t^*) & \text{if } H_1: \mu_X > \mu_Y \\ \mathbb{P}_{H_0}(T \leq t^*) & \text{if } H_1: \mu_X < \mu_Y \end{cases}$$

Two-sample t ~~st~~ test: $\underbrace{\sigma_X^2 = \sigma_Y^2}_{\sigma^2 \triangleq}$ (additional assumption)

$$\text{pooled sample variance} = \frac{(n-1)S_X^2 + (m-1)S_Y^2}{n+m-2} = \frac{\sum_{i=1}^{n}(X_i - \bar{X}_n)^2 + \sum_{i=1}^{m}(Y_i - \bar{Y}_m)^2}{n+m-2}$$

Recall: $\frac{(n-1)S_X^2}{\sigma^2} \sim \chi^2_{n-1}$, $\frac{(m-1)S_Y^2}{\sigma^2} \sim \chi^2_{m-1}$, $S_X^2 \perp S_Y^2$

$$\Rightarrow \frac{(n-1)S_X^2 + (m-1)S_Y^2}{\sigma^2} \sim \chi^2_{n+m-2} \quad (1)$$

combined w/ $$\frac{\bar{X}_n - \bar{Y}_n}{\sqrt{\frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}}} \overset{H_0}{\sim} N(0,1)$$

$\| \ \sigma_X^2 = \sigma_Y^2$

$$\frac{\bar{X}_n - \bar{Y}_n}{\sigma\sqrt{\frac{1}{n} + \frac{1}{m}}} \overset{H_0}{\sim} N(0,1) \quad (2)$$

Based on the def of t distribution

(1) + (2) $\Rightarrow$

$\bar{X}_n \perp S_X^2$

$\bar{Y}_n \perp S_Y^2$

$$\frac{\frac{\bar{X}_n - \bar{Y}_n}{\sigma\sqrt{\frac{1}{n} + \frac{1}{m}}}}{\sqrt{\left(\frac{(n-1)S_X^2 + (m-1)S_Y^2}{\sigma^2}\right)/(n+m-2)}} \overset{H_0}{\sim} t_{n+m-2}$$

$\|$

$$\frac{\bar{X}_n - \bar{Y}_n}{\sqrt{\frac{(n-1)S_X^2 + (m-1)S_Y^2}{n+m-2}\left(\frac{1}{n} + \frac{1}{m}\right)}} = \boxed{\frac{\bar{X}_n - \bar{Y}_n}{\sqrt{S_p^2\left(\frac{1}{n} + \frac{1}{m}\right)}}} \overset{H_0}{\sim} t_{n+m-2}$$

$T \ \|$ "two-sample t statistic"

Neyman - Pearson Lemma :

$H_0: \theta = \theta_0$ vs. $H_1: \theta = \theta_1 \neq \theta_0$

$X_1, \cdots, X_n \overset{iid}{\sim} f_\theta$

$L(\theta_0) = \prod_{i=1}^{n} f_{\theta_0}(X_i)$ $\qquad$ $L(\theta_1) = \prod_{i=1}^{n} f_{\theta_1}(X_i)$

The most powerful test statistic : $\dfrac{L(\theta_1)}{L(\theta_0)}$

The rejection region is $\dfrac{L(\theta_1)}{L(\theta_0)} \geq a$ where a is a threshold determined by sig. level α and the dist of $\dfrac{L(\theta_1)}{L(\theta_0)}$ under H_0