b) Use the t Table (CDF and quantiles of t dist w/ df) to find the 80th percentile of the $F_{1,30}$ distribution

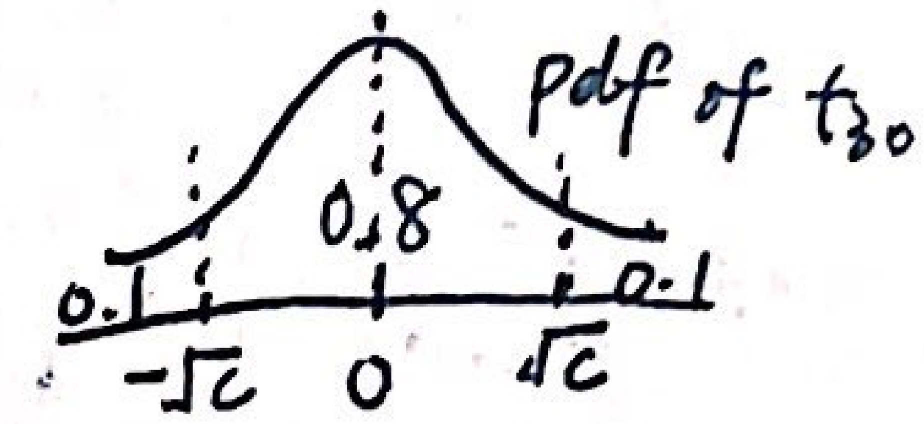
Ans: Let $T \sim t_{30}$, then $T^2 \sim F_{1,30}$

We need to find c s.t. $P(T^2 \le c) = 0.8$, $c > 0$

Since $P(T^2 \le c) = P(-\sqrt{c} \le T \le \sqrt{c})$,

so $\sqrt{c}$ is the 90% percentile of t_{30}. qt(0.9, 30)

By the CDF of t_{30}, $\sqrt{c} = 1.31 \Rightarrow c = 1.7161$



c) Use the CDF of $F_{20,60}$ to find $c > 0$ s.t. $P(X > c) = 0.99$ where $X \sim F_{60,20}$

Ans: Given $X \sim F_{60,20}$, then $\frac{1}{X} \sim F_{20,60}$

$$P(X > c) = P\left(\frac{1}{X} < \frac{1}{c}\right) = 0.99$$

So $\frac{1}{c}$ is the 99% percentile of $F_{20,60}$ qf(0.99, df1=20, df2=60)

By the CDF of $F_{20,60}$, $\frac{1}{c} \approx 2.20 \Rightarrow c \approx 0.4545$

Properties of estimators:

$\underbrace{X_1, \cdots, X_n}_{\text{random sample w/ size } n}$ iid r.v.s $\overset{\text{random}}{\sim}_{\text{sampling}}$ population with parameter θ

estimator $\hat{\theta} = f(X_1, \cdots, X_n)$ is a r.v.

Our actual data is a realization of the random sample: $\underset{0.1}{\overset{\shortparallel}{x_1}}, \cdots, \underset{1.2}{\overset{\shortparallel}{x_n}}$

$\Rightarrow$ a realization of $\hat{\theta}$

Property 1: bias of $\hat{\theta}$

$\text{bias}(\hat{\theta}) = \underbrace{E[\hat{\theta}]}_{\text{the average of all possible realizations of } \hat{\theta}} - \theta$ If $\text{bias}(\hat{\theta}) = 0$, we call $\hat{\theta}$ an <u>unbiased</u> estimator

Theorem: $X_1, \cdots, X_n \overset{iid}{\sim}$ dist w/ mean μ, variance σ^2

Then $\overline{X}_n$ is an unbiased estimator of μ, and S_n^2 is an unbiased estimator of σ^2.

Pf. $\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$, $S_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X}_n)^2$

$E[\overline{X}_n] = E\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{1}{n}\sum_{i=1}^{n} E[X_i] = \frac{1}{n}\sum_{i=1}^{n}\mu = \mu \Rightarrow \overline{X}_n$ is unbiased for μ

$E[S_n^2] = \frac{1}{n-1}\sum_{i=1}^{n} E[(X_i - \overline{X}_n)^2]$ $\underline{X_1, \cdots, X_n \text{ iid}} \Rightarrow E[(X_1 - \overline{X}_n)^2] = \cdots = E[(X_n - \overline{X}_n)^2]$

$= \frac{1}{n-1}\sum_{i=1}^{n} E[(X_1 - \overline{X}_n)^2] = \frac{n}{n-1} E[(X_1 - \overline{X}_n)^2]$

$E[(X_1-\bar{X}_n)^2] = E[X_1^2 - 2X_1\bar{X}_n + (\bar{X}_n)^2] = E[X_1^2] - 2E[X_1\bar{X}_n] + E[(\bar{X}_n)^2]$

$E[X_1^2] = (E[X_1])^2 + var[X_1] = \mu^2 + \sigma^2$ (1)

Since $E[\bar{X}_n] = \mu$ and $var[\bar{X}_n] = \underbrace{\frac{1}{n} var[X_1]}_{\frac{1}{n^2}\sum_{i=1}^{n} var[X_i] =} = \frac{\sigma^2}{n}$

so $E[(\bar{X}_n)^2] = (E[\bar{X}_n])^2 + var[\bar{X}_n] = \mu^2 + \frac{\sigma^2}{n}$ (2)

Since $X_1\bar{X}_n = X_1\left(\frac{X_1 + \cdots + X_n}{n}\right) = \frac{X_1^2}{n} + \frac{X_1X_2 + \cdots + X_1X_n}{n}$

so $E[X_1\bar{X}_n] = \frac{1}{n}E[X_1^2] + \frac{1}{n}\sum_{i=2}^{n} E[X_1X_i] = \frac{1}{n}(\mu^2+\sigma^2) + \frac{1}{n}\cdot(n-1)\cdot E[X_1X_2]$

$\overset{X_1,X_2\text{ indep}}{=} \frac{1}{n}(\mu^2+\sigma^2) + \frac{n-1}{n}(\underbrace{E[X_1]\cdot E[X_2]}_{=\mu^2}) = \mu^2 + \frac{1}{n}\sigma^2$ (3)

Hence, $E[(X_1-\bar{X}_n)^2] = \mu^2+\sigma^2 - 2(\mu^2 + \frac{1}{n}\sigma^2) + \mu^2 + \frac{\sigma^2}{n}$

$= (1 - \frac{2}{n} + \frac{1}{n})\sigma^2 = \frac{n-1}{n}\sigma^2$

So $E[S_n^2] = \frac{n}{n-1}E[(X_1-\bar{X}_n)^2] = \frac{n}{n-1}\cdot\frac{n-1}{n}\sigma^2 = \sigma^2 \Rightarrow S_n^2$ is unbiased

So $E\left[\frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X}_n)^2\right] = E\left[\frac{(n-1)S_n^2}{n}\right] = \frac{n-1}{n}\sigma^2 \Rightarrow$ biased

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Method of moments (MoM) estimators — 1^{st} way of constructing estimators

Use sample moments as estimators of population moments

$X_1, \cdots, X_n$ random sample population $\quad X \sim$ population

	sample moments		population moments
1^{st}	$\frac{1}{n}\sum_{i=1}^{n} X_i = \bar{X}_n$	unbiased $\longrightarrow$	$E[X] = \mu$
2^{nd}	$\frac{1}{n}\sum_{i=1}^{n} X_i^2$		$E[X^2] = \mu^2 + \sigma^2$
$\vdots$	$\vdots$		$\vdots$
k-th	$\frac{1}{n}\sum_{i=1}^{n} X_i^k$		$E[X^k]$

So the MoM estimator of $E[X^k]$ is $\frac{1}{n}\sum_{i=1}^{n} X_i^k$

If $\theta = g(E[X], E[X^2])$ for example, the $\hat{\theta}_{MOM} = g(\frac{1}{n}\sum_{i=1}^{n} X_i, \frac{1}{n}\sum_{i=1}^{n} X_i^2)$

Examples: 1. $X_1, \cdots, X_n \overset{iid}{\sim} \text{Poisson}(\lambda)$. Find the $\hat{\lambda}_{MOM}$.

Ans: Let $X \sim \text{Poisson}(\lambda)$. So $E[X] = \lambda$

so $\hat{\lambda}_{MOM} = \bar{X}_n$

2. $X_1, \cdots, X_n \sim \text{Unif}(0,\theta)$, Find $\hat{\theta}_{MOM}$.

Ans: Let $X \sim \text{Unif}(0,\theta)$. Then $E[X] = \frac{\theta}{2} \Rightarrow \theta = 2E[X]$

So $\hat{\theta}_{MOM} = 2\overline{X}_n$

3. $X_1, \cdots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$. Find $\hat{\mu}_{MOM}$ and $\hat{\sigma}^2_{MOM}$.

Ans: Let $X \sim N(\mu, \sigma^2)$. Then $E[X] = \mu$, $E[X^2] = \mu^2 + \sigma^2$

So
$$\begin{cases} \mu = E[X] \\ \sigma^2 = E[X^2] - (E[X])^2 \end{cases} \Rightarrow \begin{cases} \hat{\mu}_{MOM} = \overline{X}_n \\ \hat{\sigma}^2_{MOM} = \frac{1}{n}\sum_{i=1}^n X_i^2 - (\overline{X}_n)^2 \end{cases}$$

We can show that
$$\hat{\sigma}^2_{MOM} = \frac{1}{n}\sum_{i=1}^n (X_i - \overline{X}_n)^2 = \frac{1}{n}\sum_{i=1}^n \left(X_i^2 - 2X_i\overline{X}_n + (\overline{X}_n)^2\right)$$
$$= \frac{1}{n}\Big[\sum X_i^2 - 2\overline{X}_n \underbrace{(\sum X_i)}_{=n\overline{X}_n} + n(\overline{X}_n)^2\Big]$$
$$= \frac{1}{n}\left[\sum X_i^2 - n(\overline{X}_n)^2\right]$$

So $\hat{\sigma}^2_{MOM}$ is a biased estimator of σ^2.

4. $X_1, \cdots, X_n \overset{iid}{\sim} N(0, \sigma^2)$. Find $\hat{\sigma}^2_{MOM}$.

Ans: Let $X \sim N(0,\sigma^2)$. Then $E[X] = 0$, $E[X^2] = (E[X])^2 + Var[X] = \sigma^2$

So $\hat{\sigma}^2_{MOM} = \frac{1}{n}\sum_{i=1}^n X_i^2 = \frac{1}{n}\sum_{i=1}^n (X_i - 0)^2$

5. $X_1, \cdots, X_n$ iid w/ pdf $f(x) = (\theta+1)x^\theta$, $0 < x < 1$, $\theta > -1$. Find $\hat{\theta}_{MOM}$.

Ans: Let $X \sim f$. Then
$$E[X] = \int_0^1 x f(x)\,dx = \int_0^1 x \cdot (\theta+1)x^\theta\,dx$$
$$= \int_0^1 (\theta+1)x^{\theta+1}\,dx = \frac{\theta+1}{\theta+2}x^{\theta+2}\Big|_0^1 = \frac{\theta+1}{\theta+2}.$$

$$\Rightarrow \theta = \frac{2E[X]-1}{1-E[X]}.$$

$$\Rightarrow \hat{\theta}_{MOM} = \frac{2\overline{X}_n - 1}{1 - \overline{X}_n}.$$