X : r.v. domain : $\mathcal{X}$ e.g. $\mathcal{X} = \mathbb{R} = (-\infty, +\infty)$

Pick $x \in \mathcal{X}$ e.g. $x = 0$

$$0 \le \mathbb{P}(X = \underline{x}) \le 1 = \begin{cases} 0 & \text{if } X \text{ is } \cancel{\text{discrete}} \text{ continuous} \\ \ge 0 & \text{discrete} \quad \text{probability mass function (pmf)} \end{cases}$$

if X is continuous : probability density function (pdf) : $f(x)$

$$\mathbb{P}(a \le X \le b) = \mathbb{P}(a < X \le b) = \mathbb{P}(a \le X < b) = \mathbb{P}(a < X < b)$$

$$= \int_a^b f(x)\,dx = \int_a^b f(t)\,dt = E[\underbrace{I(a < X < b)}_{\sim \text{Bernoulli}(\mathbb{P}(a<X<b))}]$$

- $f(x) \ge 0 \quad \forall x \in \mathcal{X}$
- $\int_{\mathcal{X}} f(x)\,dx \overset{\text{e.g.}}{=} \int_{-\infty}^{+\infty} f(x)\,dx = 1$

cumulative distribution function (CDF)

$$\mathbb{P}(X \le x) = \begin{cases} \sum_{y \le x} \underbrace{\mathbb{P}(X = y)}_{\text{pmf of } X \text{ at } y} & \text{if } X \text{ is discrete} \\ \int_{-\infty}^{x} f(y)\,dy \end{cases}$$

Moments :

1st moment = mean = expectation $= E[X] = EX = \Big\{$

$$= \begin{cases} \sum_{x \in \mathcal{X}} x\, \mathbb{P}(X = x) & \text{if } X \text{ is discrete} \\ \int_{\mathcal{X}} x f(x)\,d\cancel{x} & \text{continuous} \end{cases}$$

$$2^{nd} \text{ moment} = E[X^2] = \begin{cases} \sum_{\mathcal{X}} x^2\, \mathbb{P}(X = x) \\ \int_{\mathcal{X}} x^2 f(x)\,dx \end{cases}$$

Moment generating function (MGF)

X r.v. $\in \mathbb{R}$

$\forall t \in \mathbb{R}$, define the MGF of X is

$$M_X(t) = E[e^{tX}] = \begin{cases} \sum_x e^{tx} \cdot P(X=x) & \text{if } X \text{ is discrete} \\ \int_{-\infty}^{+\infty} e^{tx} \underbrace{f(x)}_{\text{pdf}} dx & \text{if } X \text{ is cont.} \end{cases}$$

Taylor expansion on e^{tX}:

$$e^{tX} = 1 + \frac{tX}{1!} + \frac{(tX)^2}{2!} + \frac{(tX)^3}{3!} + \cdots$$

$$\underset{M_X(t)}{\overset{\shortparallel}{E[e^{tX}]}} = 1 + \frac{tE[X]}{1!} + \frac{t^2 E[X^2]}{2!} + \frac{t^3 E[X^3]}{3!} + \cdots \quad (1)$$

Taylor expansion of $M_X(t)$:

$$M_X(t) = \underbrace{M_X(0)}_{=1} + \frac{t \cdot M_X'(0)}{1!} + \frac{t^2 \cdot M_X''(0)}{2!} + \frac{t^3 \cdot M_X'''(0)}{3!} + \cdots \quad (2)$$

$$(1) = (2) \iff \begin{cases} E[X] = M_X'(0) \\ E[X^2] = M_X''(0) \\ E[X^3] = M_X'''(0) \\ \vdots \end{cases} \iff E[X^k] = M_X^{(k)}(0)$$

$k = 1, 2, \cdots$

if $M_X^{(k)}(t)$ exists at an interval around 0.

Example: ① $X \sim$ Binomial (n, p)

$$P(X = x) = \binom{n}{x} p^x (1-p)^{n-x} \quad x = 0, 1, \cdots, n$$

$$M_X(t) = E[e^{tX}] = \sum_{x=0}^{n} e^{tx} P(X=x) = \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^x (1-p)^{n-x}$$

$$= \sum_{x=0}^{n} \binom{n}{x} (e^t \cdot p)^x \cdot (1-p)^{n-x}$$

$$\boxed{(a+b)^n = \sum_{x=0}^{n} \binom{n}{x} a^x b^{(n-x)}}$$

$$= (e^t p + 1 - p)^n$$

Know $E[X] = np$ $\quad var[X] = np(1-p)$

$E[X^2] = (E[X])^2 + var[X] = n^2p^2 + np(1-p) = n^2p^2 + np - np^2$

Verify $E[X]$, $E[X^2]$ by $M_X(t)$:

$M_X'(t) = n(e^t p + 1 - p)^{n-1} \cdot e^t p \Rightarrow M_X'(0) = n \cdot p.$

$M_X''(t) = n(n-1)(e^t p + 1 - p)^{n-2} \cdot e^t p \cdot e^t p + n(e^t p + 1 - p)^{n-1} \cdot e^t p.$

$\Rightarrow M_X''(0) = n(n-1)p^2 + np = n^2p^2 - np^2 + np.$

② $X \sim \text{Poisson}(\lambda)$ $\quad \lambda > 0$

$\mathbb{P}(X = x) = \dfrac{\lambda^x \cdot e^{-\lambda}}{x!}$, $x = 0, 1, 2, \ldots$ $\quad \sum_{x=0}^{\infty} \mathbb{P}(X=x) = 1$

$= \sum_{x=0}^{\infty} \dfrac{\lambda^x e^{-\lambda}}{x!} = 1$

$M_X(t) = \sum_{x=0}^{\infty} e^{tx} \cdot \mathbb{P}(X = x)$